New Non-Diagonal Singularity-Free Cosmological Perfect-Fluid Solution

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Abstract

We present a new non-diagonal G_2 inhomogeneous perfect-fluid solution with barotropic equation of state $p = \rho$ and positive density everywhere. It satisfies the global hyperbolicity condition and has no curvature singularity anywhere. This solution is very simple in form and has two arbitrary constants.

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The study of exact inhomogeneous cosmological models is mainly motivated by the fact that our Universe is not strictly spatially homogeneous. It was extensively believed until very recently that all the cosmological models (homogeneous or not) had to originate in a big-bang singularity whenever they satisfied an energy condition for the matter contents and the model contained matter everywhere. This belief was greatly reinforced by the powerful singularity theorems [1]. However, in a remarkable letter published by J.M.M. Senovilla in 1990 [2], a new perfect-fluid inhomogeneous cosmological solution without big-bang singularity and without any other curvature singularity was presented. The matter contents of that solution satisfied a realistic equation of state for radiation dominated era $p = \rho/3$ with density (and therefore pressure) positive everywhere.

In a subsequent paper [3] this solution was shown to be geodesically complete and satisfying causality conditions such as global hyperbolicity. It was also proven that the singularity theorems for the existence of incomplete geodesics could not be applied in that case because they need some boundary or initial conditions, such as the existence of a causally trapped set, which did not hold in that case. In consequence, the existence

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of physically well-behaved cosmological solutions regular everywhere was clearly stated without contradicting the powerful and aesthetically beautiful singularity theorems.

The above-mentioned singularity-free solution was generalized in a paper [4] where the case of G_2 diagonal cosmologies assuming separation of variables in co-moving coordinates was extensively studied. Among the huge variety of perfect-fluid solutions with two commuting and mutually orthogonal Killing vectors satisfying an Ansatz of separation of variables in a coordinate system adapted to the fluid velocity vector, all the different singular behaviours were possible; there were solutions with a big-bang and/or big-crunch singularity, solutions with timelike singularities but free of initial or final singularities, solutions with pressure and density regular everywhere but with singularities in the Weyl tensor, and finally a family of solutions completely free of singularities of any type which contained the $p = \rho/3$ as a particular case. It was also shown that this family was the most general solution without singularities under the stated assumptions.

There have been some very recent attempts to prove that this family of singularity-free solutions is unique in some sense. In fact, N. Dahdich & L. K. Patel [5] may have proven that for diagonal cosmologies

$$ds^2 = -A_0^2 dt^2 + A_1^2 dx_1^2 + A_2^2 dx_2^2 + A_3^2 dx_3^2$$

with fluid velocity vector $\vec{u}$ parallel to ∂_t (adapted coordinates) and each of the functions A_α being a product of a function of only t with a function of only the spatial coordinates (separation of variables) but without assuming any symmetries in the spacetime, the only singulariry-free solution is given by the family pressented in [4] with a two-dimensional group of isometries.

The purpose of this letter is to present a new perfect-fluid cosmological solution of Einstein's equations without big-bang singularity or any other curvature singularities, neither in the pressure or density nor in the Weyl tensor, satisfying a physically well-motivated equation of state for a stiff fluid $p = \rho$, with density positive and non-vanishing everywhere and satisfying a causality condition, namely global hyperbolicity. This solution is very simple in form, and possesses a two-dimensional abelian group of isometries acting on spacelike surfaces, but with neither of the Killing vectors being hypersurface-orthogonal. Therefore, the metric is non-diagonal and it belongs to the class B(i) in Wainwright's classification of G_2 cosmologies [6].

The line-element of this solution is given by

$$ds^2 = e^{\alpha a^2 r^2} \cosh(2at) \left(-dt^2 + dr^2\right) + r^2 \cosh(2at) d\phi^2 + \frac{1}{\cosh(2at)} \left(dz + ar^2 d\phi\right)^2 ,$$

where α and a are arbitrary constants, and the range of variation of the coordinates is

$$-\infty < t, z < \infty \qquad 0 \leq r < \infty \qquad 0 \leq \phi \leq 2\pi .$$

This spacetime has a well-defined axis of symmetry at $r = 0$ where the so-called elementary flatness [7] is safisfied and therefore the coordinate r has to be interpreted as a radial

cylindrical coordinate. The energy-momentum tensor of this spacetime corresponds to a perfect fluid with velocity vector given by

$$\vec{u} = \frac{e^{-\frac{1}{2}\alpha a^2 r^2}}{\cosh^{1/2}(2at)}\partial_t$$

and density and pressure

$$p = \rho = \frac{a^2\,(\alpha-1)\,e^{-\alpha a^2 r^2}}{\cosh(2at)},$$

which are positive everywhere whenever the constant α is restricted to

$$\alpha > 1.$$

It can be seen that this $p = \rho$ solution can be generated from a vacuum solution using the generating procedure of Wainwright, Ince & Marshman [9]. The vacuum solution that leads to this spacetime is contained in this family as the special case $\alpha = 1$.

It is trivial to check that (when the constant a is not zero) the only Killing vectors of this metric are given by ∂_z and ∂_ϕ, except in the particular case $\alpha = 0$ in which the metric has a four-dimensional group of symmetries acting on three-dimensional spacelike hypersurfaces and that contains a three-dimensional subgroup belonging to Bianchi type II. Thus, in this particular case the solution is locally rotationally symmetric.

In the natural null tetrad that can be read from the expression of the line-element, the non-vanishing components of the Weyl tensor are given by

$$\begin{aligned}
\Psi_0 &= \frac{a^2 e^{-\alpha a^2 r^2}}{\cosh^3(2at)}\left[\cosh^2(2at)(\alpha/2+1) + \alpha a r\cosh(2at)\sinh(2at) - 3 -\right. \\
&\qquad \left. i(\alpha a r\cosh(2at) + 3\sinh(2at))\right] \\
\Psi_2 &= \frac{a^2 e^{-\alpha a^2 r^2}}{\cosh^3(2at)}\left[\cosh^2(2at)(\alpha/6+1/3) - 1 - i\sinh(2at)\right] \\
\Psi_4 &= \frac{a^2 e^{-\alpha a^2 r^2}}{\cosh^3(2at)}\left[\cosh^2(2at)(\alpha/2+1) - \alpha a r\cosh(2at)\sinh(2at) - 3 -\right. \\
&\qquad \left. i(-\alpha a r\cosh(2at) + 3\sinh(2at))\right],
\end{aligned}$$

which show that this solution has Petrov type I everywhere except on the axis of symmetry where it degenerates to type D. From the expressions above for the energy density, pressure and Weyl tensor, it is obvious that this spacetime has no curvature singularities anywhere. In fact, an easy calculation shows that not only all the scalar polynomials in the Riemann tensor are regular, but all scalars based on the derivatives of Riemann are as well.

In order to see that a causality condition is satisfied, it is sufficient to note that the coordinate t in the metric is a time function, that is, it increases along every future-directed non-spacelike curve. This can be checked by computing the gradient of this function which results timelike everywhere. The existence of a time function is the necessary and sufficient condition for the stable causality condition to hold. Using a result due to R. Geroch ([8]) concerning the null geodesics and the fact that this metric is causally stable, it can be easily proven that this spacetime is in fact globally hyperbolic. A detailed study of the geodesics of this spacetime shows that the solution is geodesically complete and therefore singularity-free.

We will use the natural orthonormal tetrad of the line-element to write the components of the kinematical tensors associated with the fluid velocity vector $\vec{u}$. The expansion and the non-zero components of the shear tensor of the fluid velocity vector are given by

$$\theta = \frac{a e^{-\frac{1}{2}\alpha a^2 r^2} \sinh(2at)}{\cosh^{3/2}(2at)}, \qquad \sigma_{11} = \sigma_{22} = -\frac{\sigma_{33}}{2} = \frac{2}{3}\theta.$$

The vorticity of $\vec{u}$ vanishes obviously and its acceleration is

$$\vec{a} = \alpha a^2 r \frac{e^{-\alpha a^2 r^2}}{\cosh(2at)} \partial_r.$$

From the expressions of the pressure, density, Weyl tensor and kinematical quantities, it follows that this solution is nearly flat when $t \to -\infty$, where all these quantities tend to zero. Then, as t increases, the fluid begins to contract while the density obviously increases. This contraction lasts until $t = 0$ when the fluid begins to expand. At this time the density reaches a maximum along the world lines of the fluid and then starts to decrease. This maximum can be made arbitrarily large by fixing the constant α big enough. As $t \to \infty$ the solution tends again to a nearly flat situation. The avoidance of collapse into a singularity during the contracting era is provided by the radial gradient of pressure which produces an eternal outwards acceleration onthe fluid. Thus, the inhomogeneity of the spacetime allows it to avoid any collapse into a singularity.

The deceleration parameter can be computed from the expression for the expansion and gives

$$q = 8 - 6\frac{\cosh^2(2at)}{\sinh^2(2at)},$$

which shows that this solution has an inflationary epoch near the rebound time $t = 0$ while it is non-inflationary for the rest of its history.

Now, we will write the diagonal limit of this solution. This limit is obtained when the constant a, which essentially gives a scale to the spacetime, is zero. This limit can be made maintaining the product $\alpha a^2 = \beta$ constant, thus leading to the metric

$$ds^2 = e^{\beta r^2}\left(-dt^2 + dr^2\right) + r^2 d\phi^2 + dz^2.$$

This metric is obviously static and its matter contents is a stiff perfect-fluid with pressure and density given by

$$p = \rho = \beta e^{-\beta r^2}.$$

The non-zero components of the Weyl tensor in this particular case are

$$\Psi_0 = \Psi_4 = 3\Psi_2 = \frac{1}{2}\beta e^{-\beta r^2},$$

so that this metric is type D everywhere. In consequence, it belongs to a class due to Barnes [10]. Being static and cylindrically symmetric it also belongs to the general solution found by K.A. Bronnikov ([11]) and rediscovered by D. Kramer ([12]). Finally, the vacuum limit of this solution ($\beta = 0$) is the Minkowski spacetime.

Summing up, we have found the first non-diagonal singularity-free cosmological model satisfying physically well-behaved properties, and besides, we have shown that the existence of singularity-free models is likely to occur when more general than separable G_2 diagonal cosmologies are considered. This solution has been found by studying the perfect-fluid orthogonally transitive G_2 cosmologies with no hypersurface-orthogonal Killing vector (class B(i) in Wainwright's classification) with an Ansatz of separation variables in co-moving coordinates. This case has been in fact exhausted and the results will be published in the near future.

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